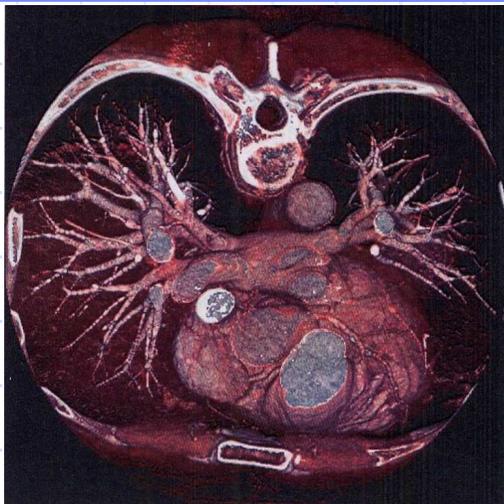


FORMATION TIC (Phymed, STIC) COMPLEMENTS EN TOMOGRAPHIE MEDICALE

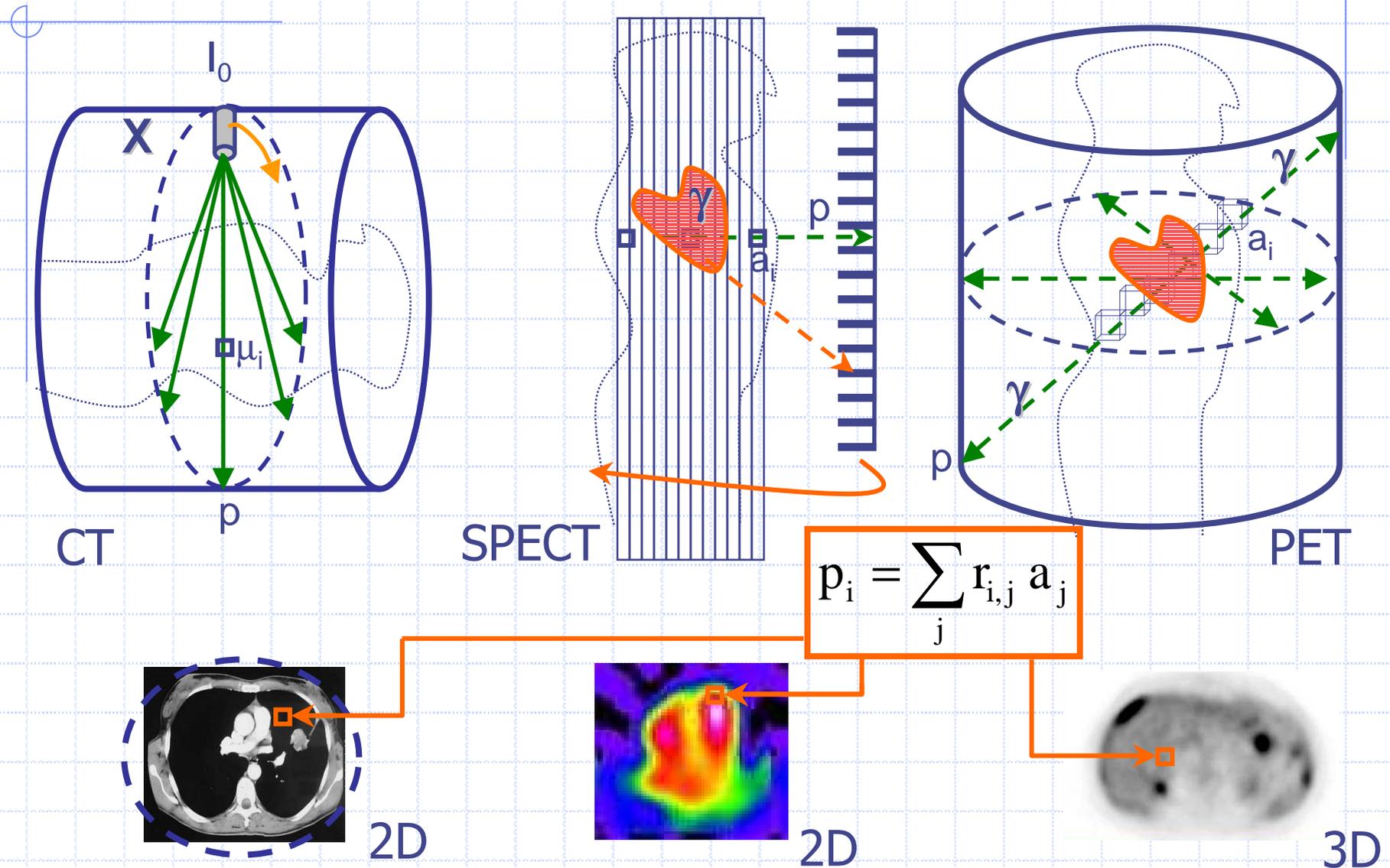


Denis Mariano-Goulart

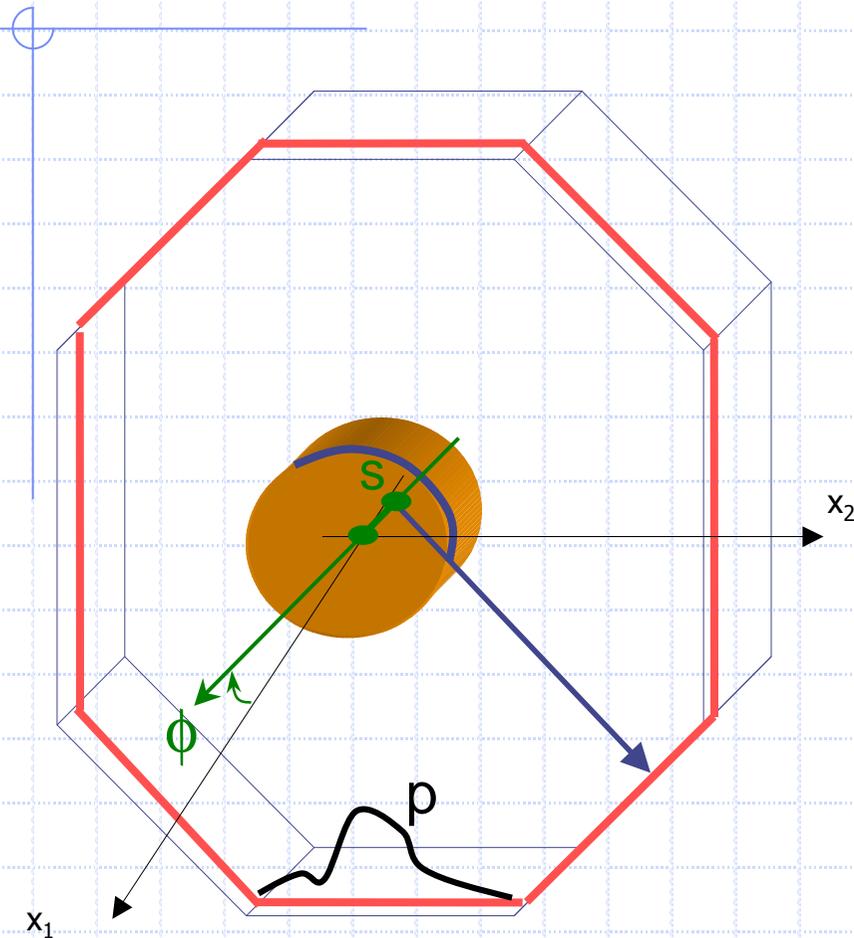
Faculté de médecine & CHRU de Montpellier

<http://scinti.etud.univ-montp1.fr>

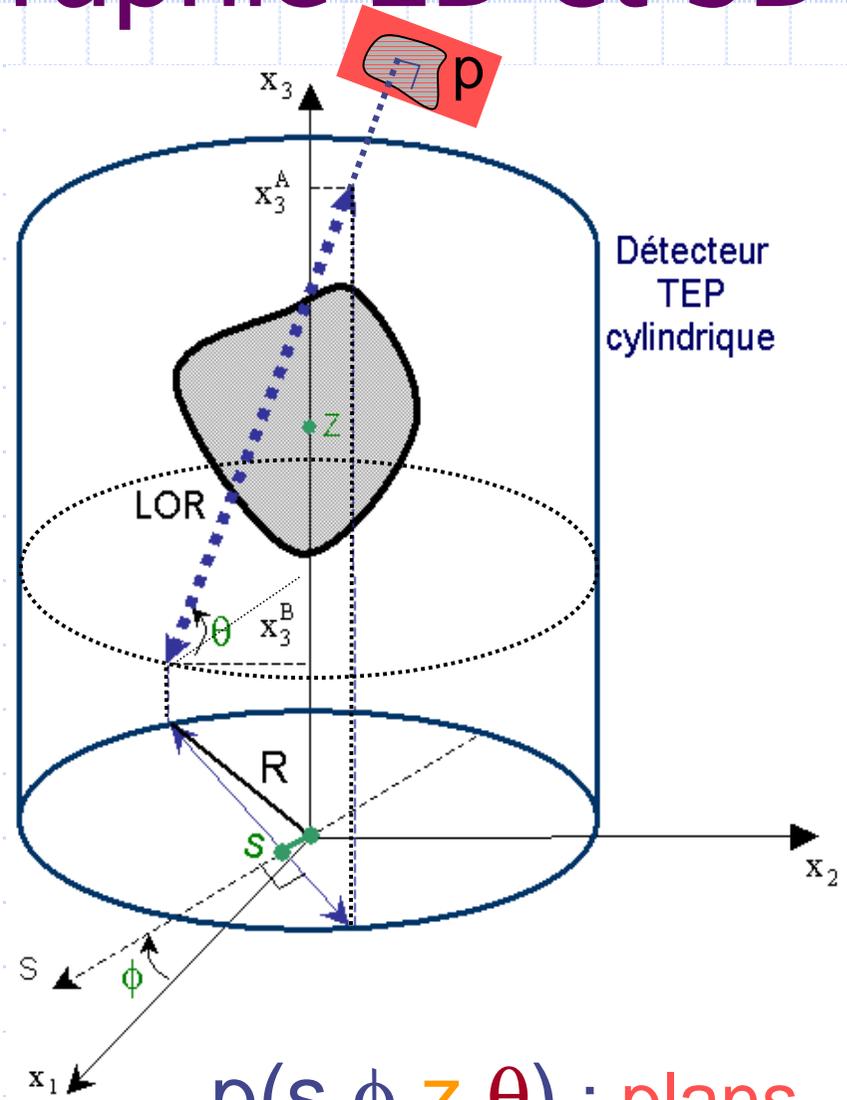
Tomographie: problème inverse linéaire



Codage en tomographie 2D et 3D

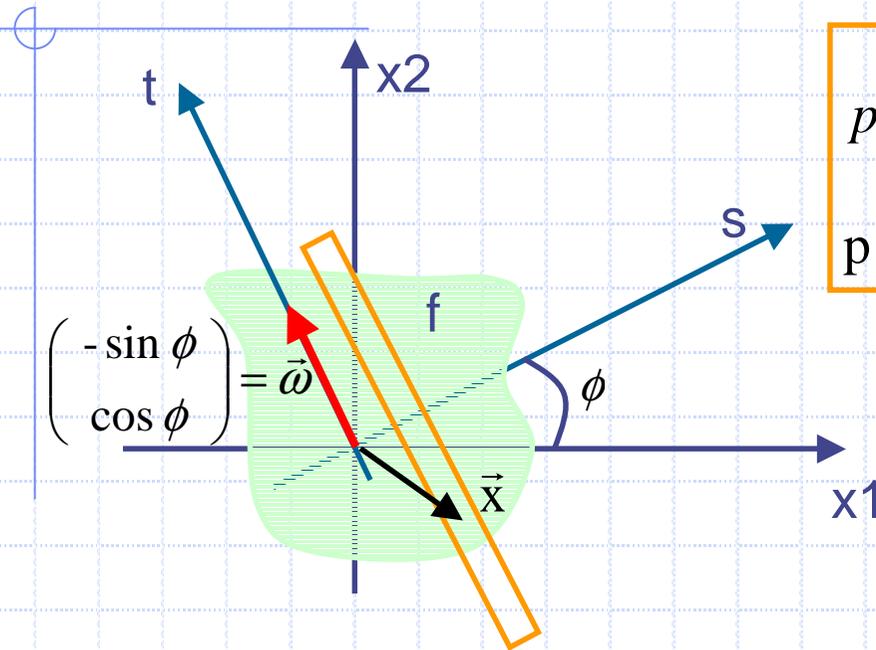


$p(s, \phi)$: lignes



$p(s, \phi, z, \theta)$: plans

Modélisation analytique

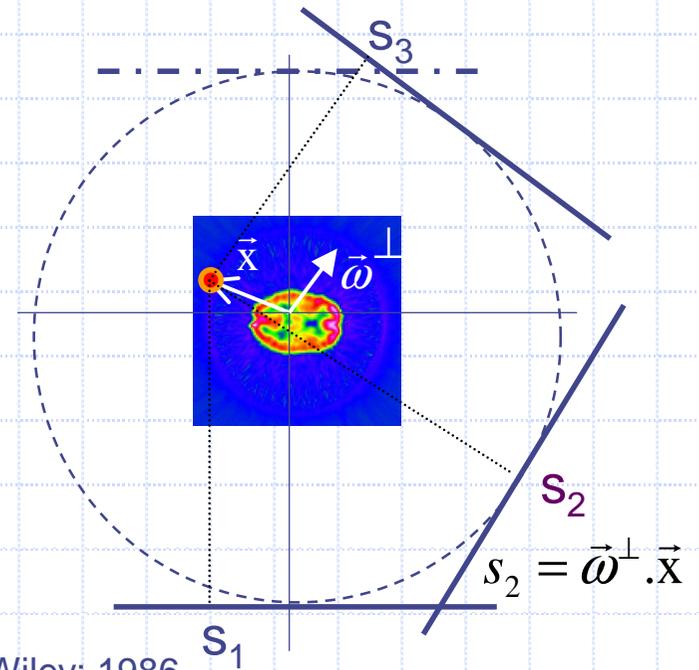


$$p(\vec{\omega}, s) = p_{\vec{\omega}}(s) = \int_t f(s \vec{\omega}^\perp + t \vec{\omega}) dt$$

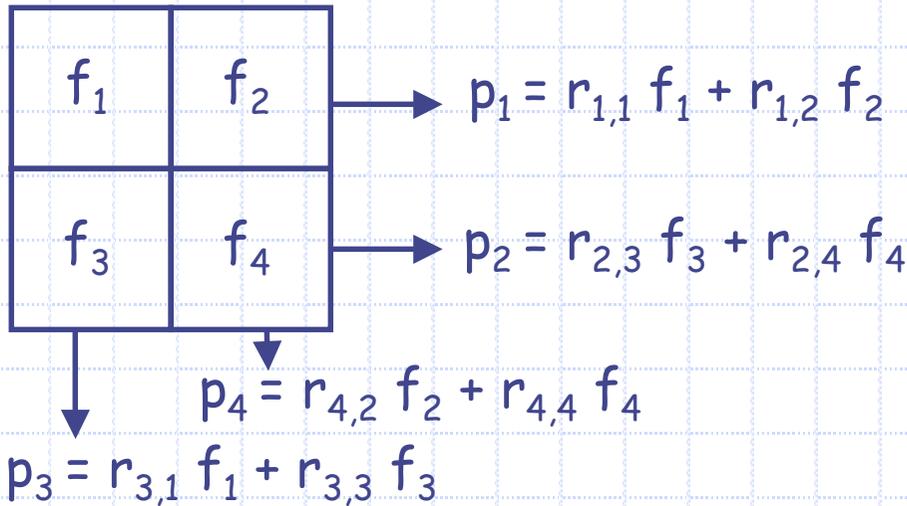
$p = Rf$ transformée de Radon

$$(R^* p)(\vec{x}) = \int_{\phi=0}^{\pi} p(\vec{\omega}, \vec{\omega}^\perp \cdot \vec{x}) d\phi$$

rétroprojection = épandage



Modélisation algébrique



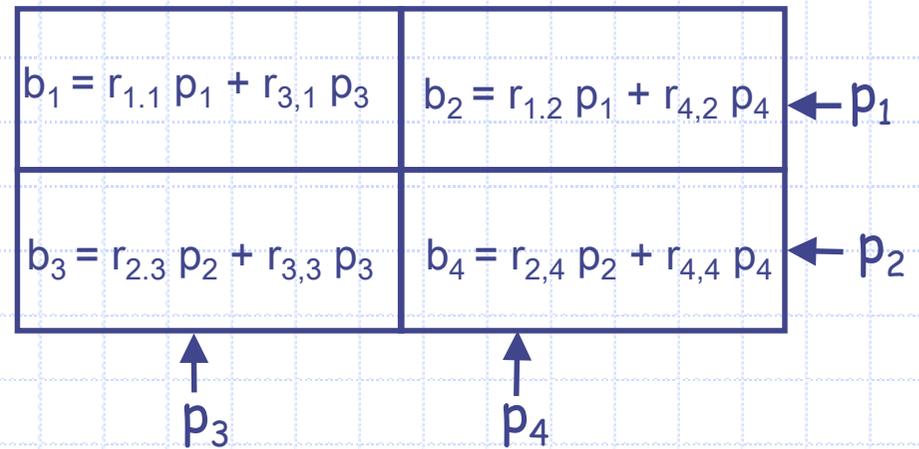
$$\begin{pmatrix} r_{1,1} & r_{1,2} & r_{1,3} & r_{1,4} \\ r_{2,1} & r_{2,2} & r_{2,3} & r_{2,4} \\ r_{3,1} & r_{3,2} & r_{3,3} & r_{3,4} \\ r_{4,1} & r_{4,2} & r_{4,3} & r_{4,4} \end{pmatrix} \cdot \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{pmatrix}$$

$r_{i,j}$ = % du pixel j intersecté par la projection i

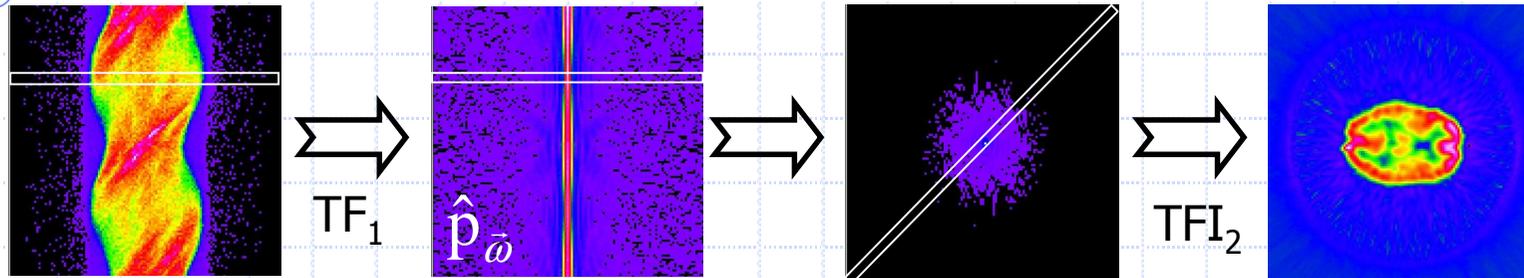
$$\mathbf{R} \cdot \vec{f} = \vec{p}$$

$$\begin{pmatrix} r_{1,1} & r_{2,1} & r_{3,1} & r_{4,1} \\ r_{1,2} & r_{2,2} & r_{3,2} & r_{4,2} \\ r_{1,3} & r_{2,3} & r_{3,3} & r_{4,3} \\ r_{1,4} & r_{2,4} & r_{3,4} & r_{4,4} \end{pmatrix} \cdot \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}$$

$${}^t\mathbf{R} \cdot \vec{p} = \vec{b}$$



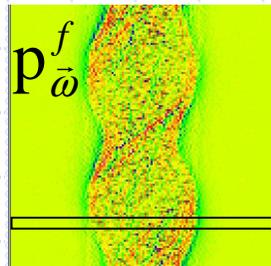
Algorithmes analytiques



$$\hat{p}_{\bar{\omega}}(\sigma) = \hat{f}(\sigma \cdot \bar{\omega}^\perp)$$

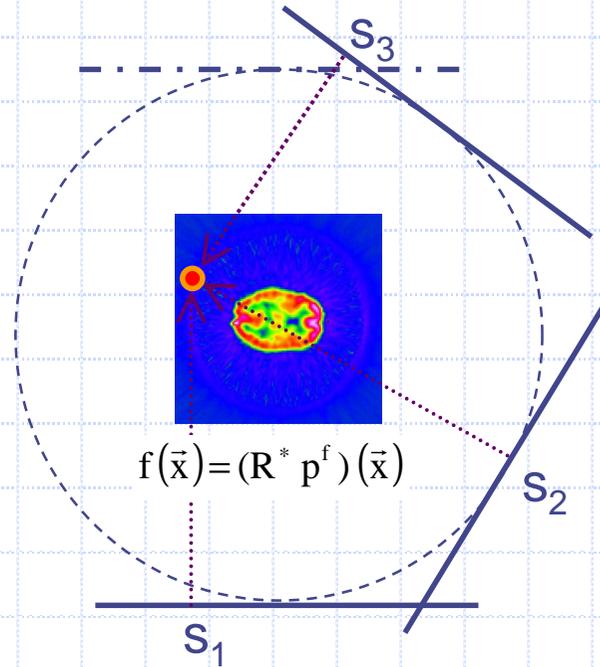
X ABS

TFI₁



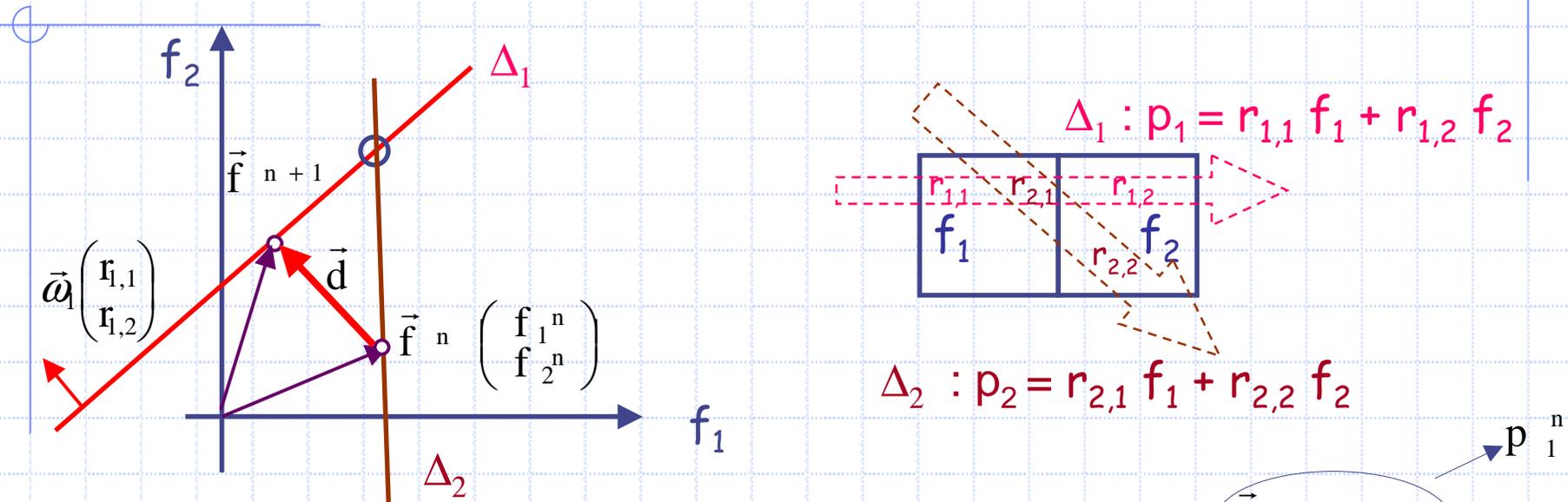
$$TF_s^{-1}[\hat{p}_{\bar{\omega}} \cdot \text{abs}] = p_{\bar{\omega}}^f$$

R*



$$f(\vec{x}) = (R^* p^f)(\vec{x})$$

Algorithmes algébriques



$$\vec{f}^{n+1} = \vec{f}^n + d \frac{\vec{\omega}_1}{\|\vec{\omega}_1\|} \quad d = \frac{p_1 - \vec{f}^n \cdot \vec{\omega}_1}{\|\vec{\omega}_1\|}$$

$$\vec{f}^{n+1} = \vec{f}^n + \frac{p_1 - p_1^n}{\|\vec{\omega}_1\|^2} \vec{\omega}_1$$

$$\vec{f}^{n+1} = \vec{f}^n + R^* (p_1 - p_1^n)$$



S. Kaczmarz
1895-1940

MLEM et OSEM

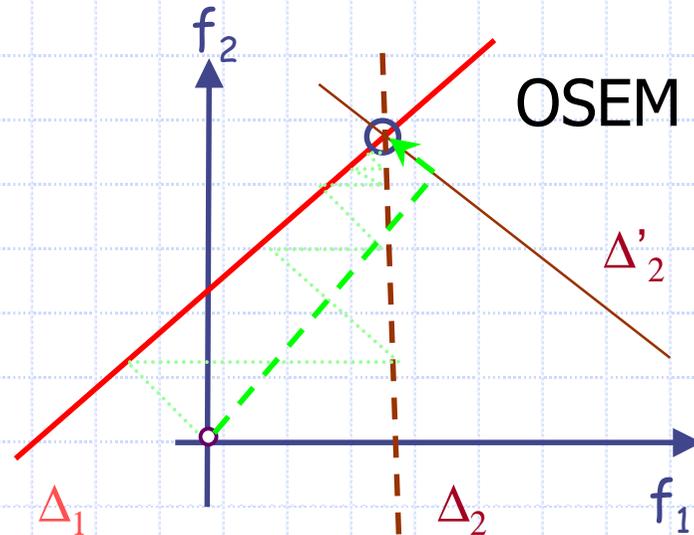
Bayes : $P(\vec{f} / \vec{p}) = P(\vec{p}/\vec{f}) \cdot P(\vec{f}) / P(\vec{p}) = P(\vec{p}/\vec{f}) \cdot P(\vec{f})$

$$\vec{f} \approx = \arg \min_{\vec{f}} \left[-\log P(\vec{p}/\vec{f}) - \log P(\vec{f}) \right]$$

Adéquation aux données

$$\log \prod_i \frac{e^{-\tilde{p}_i} \tilde{p}_i^{p_i}}{p_i!}$$

$$f_i^{n+1} = f_i^n \cdot \frac{1}{\sum_{l'=1}^P r_{l',i}} \left(\sum_{l=1}^P r_{l,i} \frac{p_l}{\sum_{s=1}^N r_{l,s} f_s^n} \right) \mathbf{R}^* \left[\frac{p_l}{p_l^n} \right]$$

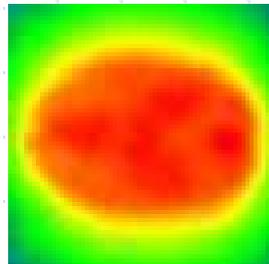


Gradient Conjugué

$$\bar{f} = \arg \min_{f \in C} \left\| R\vec{f} - \vec{p} \right\|^2$$

Initialisation :

$$\vec{d}^0 = \vec{r}^0 = R^* \cdot \vec{p} \longrightarrow$$



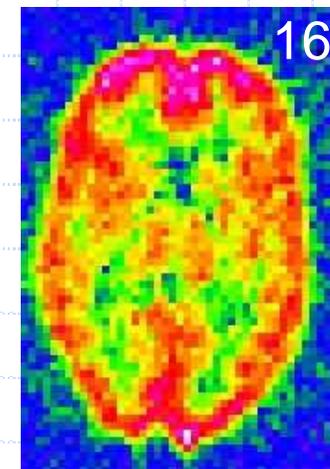
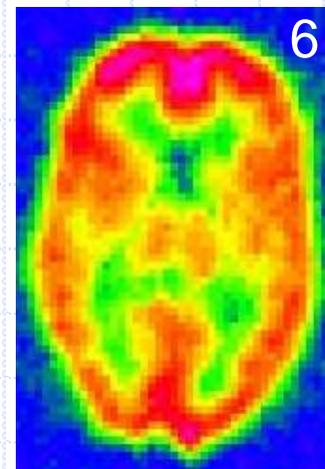
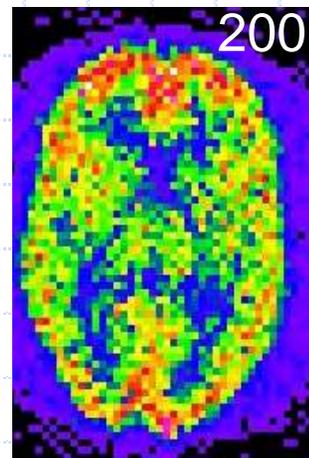
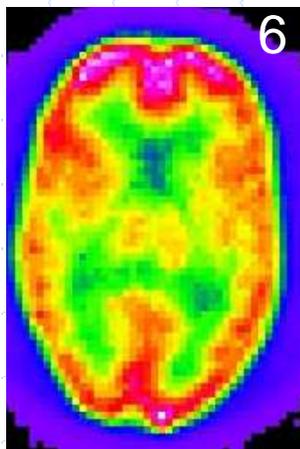
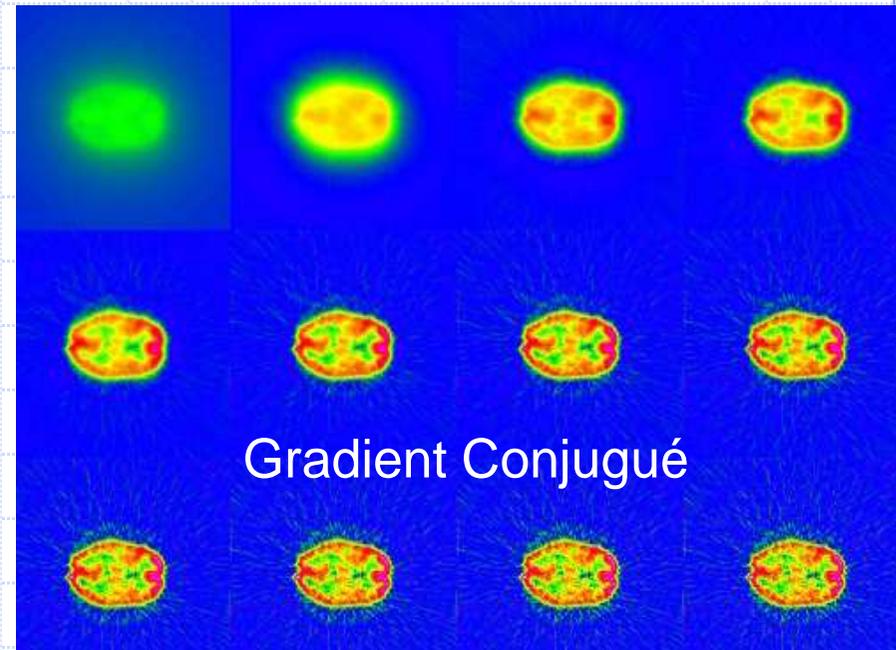
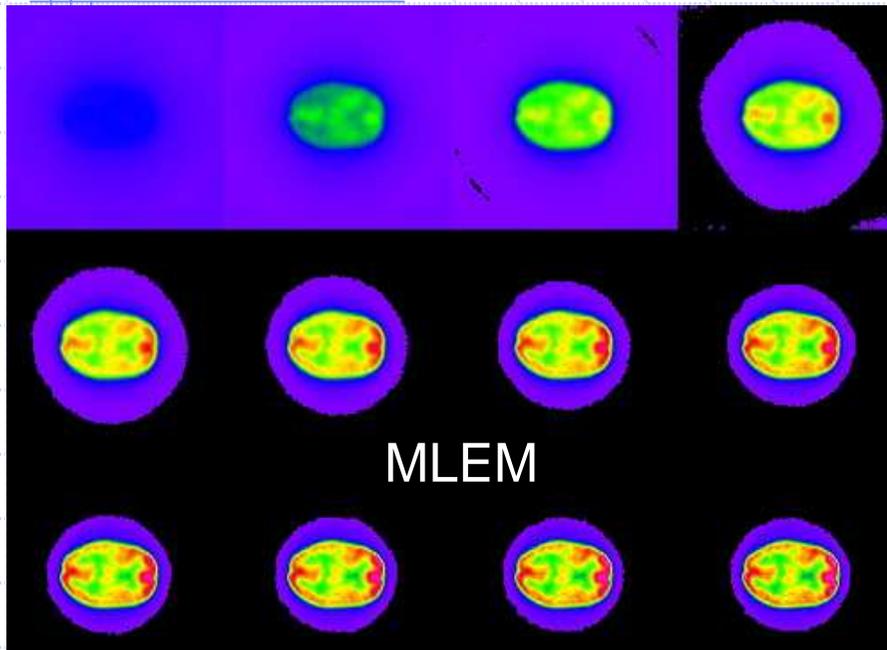
$$\omega^j = \frac{\left\| \vec{r}^j \right\|^2}{\left\langle \vec{d}^j \mid R^* \cdot R \cdot \vec{d}^j \right\rangle}$$

$$\vec{r}^{j+1} = \vec{r}^j - \omega^j \cdot R^* \cdot R \cdot \vec{d}^j$$

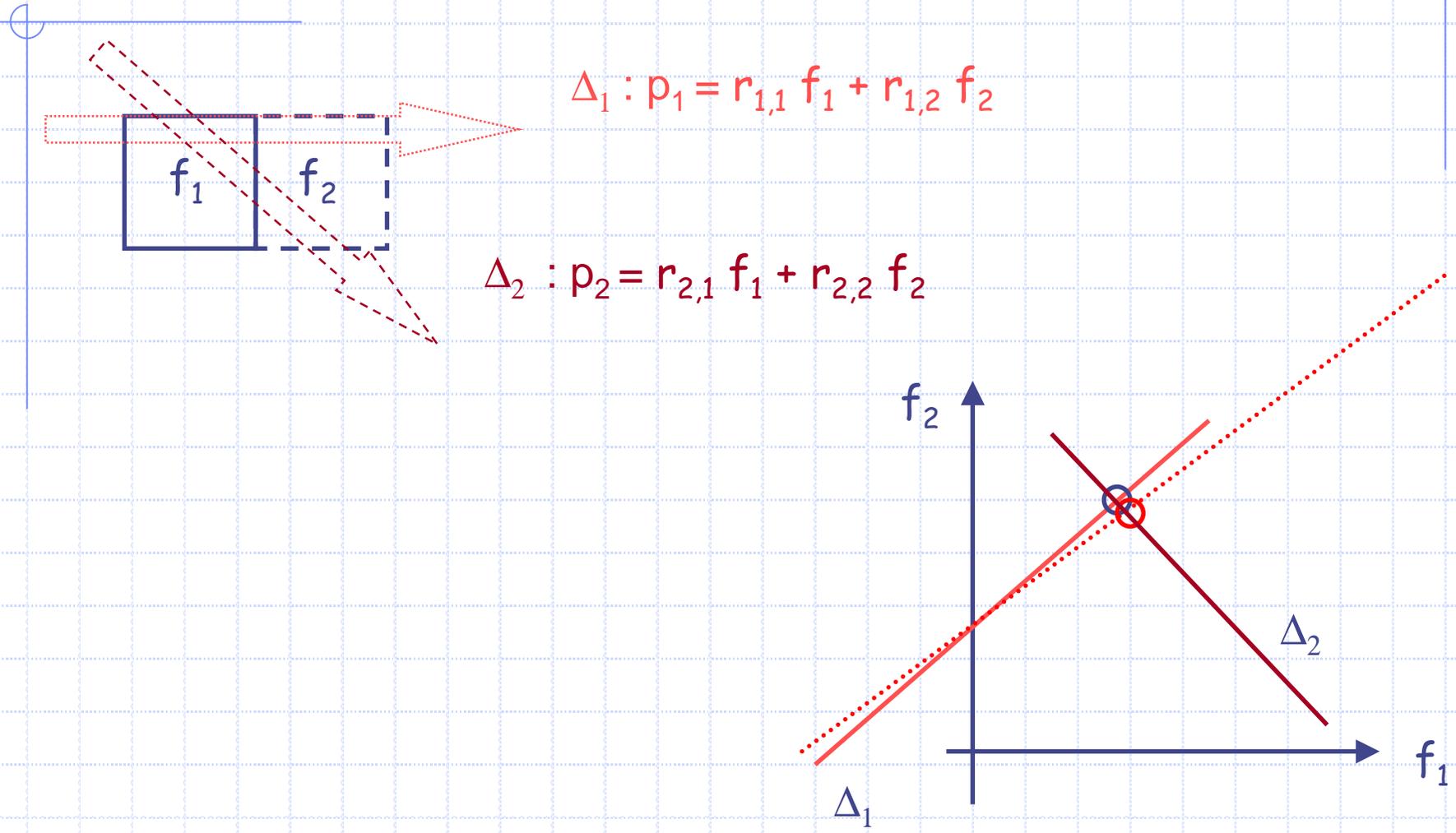
$$\vec{d}^{j+1} = \vec{r}^{j+1} + \frac{\left\| \vec{r}^{j+1} \right\|^2}{\left\| \vec{r}^j \right\|^2} \cdot \vec{d}^j$$

$$\vec{f}^{j+1} = \vec{f}^j + \omega^j \cdot \vec{d}^j$$

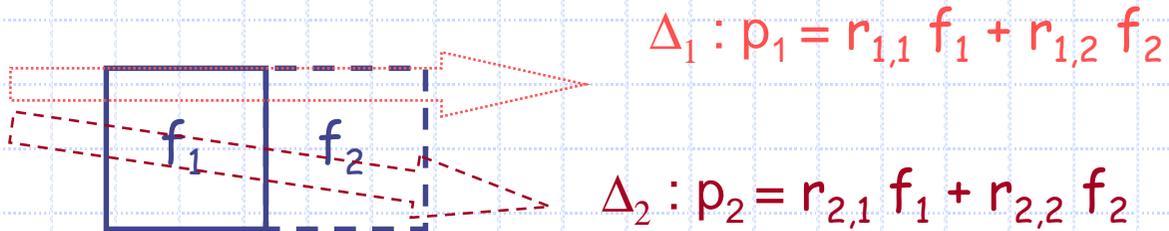
Exemples



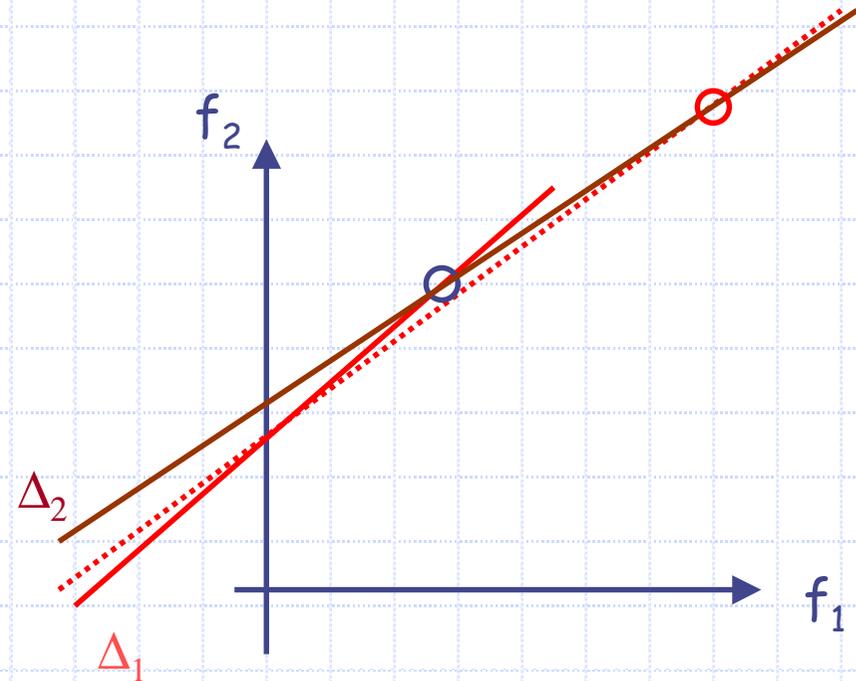
Approche intuitive (I)



Approche intuitive (II)



$64^2 = 4\ 096$
 $128^2 = 16\ 384$
 $256^2 = 65\ 536$
 $512^2 = 262\ 144$



Exemple

$$R \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 10 & 7 & 8 & 7 \\ 7 & 5 & 6 & 5 \\ 8 & 6 & 10 & 9 \\ 7 & 5 & 9 & 10 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 32 \\ 23 \\ 33 \\ 31 \end{pmatrix}$$

f p

Matrice de Wilson

Det = 1

$$\begin{pmatrix} 10 & 7 & 8 & 7 \\ 7 & 5 & 6 & 5 \\ 8 & 6 & 10 & 9 \\ 7 & 5 & 9 & 10 \end{pmatrix} \begin{pmatrix} 9,2 \\ -12,6 \\ 4,5 \\ -1,1 \end{pmatrix} = \begin{pmatrix} 32,1 \\ 22,9 \\ 33,1 \\ 30,9 \end{pmatrix} \qquad \begin{pmatrix} 10 & 7 & 8 & 7 \\ 7 & 5 & 6 & 5 \\ 8 & 6 & 10 & 9 \\ 7 & 5 & 9 & 10 \end{pmatrix} \begin{pmatrix} -7,2 \\ 14,6 \\ -2,5 \\ 3,1 \end{pmatrix} = \begin{pmatrix} 31,9 \\ 23,1 \\ 32,9 \\ 31,1 \end{pmatrix}$$

$$\kappa(R) = \|R\| \cdot \|R^{-1}\| = \frac{\mu_{\max}}{\mu_{\min}}$$

où $\mu = \sqrt{\text{valeurs propres de } R^t R}$

$$\text{Sp}(R) \approx \{0,01; 0,84; 3,86; 30,29\} \Rightarrow \kappa(R) \approx \frac{30,29}{0,01} \approx 3029$$

La matrice de Wilson est très mal conditionnée ($\kappa \gg 1$)

Problème d'Hadamard bien posé ?

✓ En continu : $\hat{p}_{\vec{\omega}}(\sigma) = \hat{f}(\sigma \cdot \vec{\omega}^\perp)$, R bijectif d'inverse continue (conditions d'Hadamard).

✓ En discret, les choses sont moins simples :

- R surjectif ? $\Leftrightarrow {}^t\mathbf{R} \cdot \mathbf{R} \vec{f} = \mathbf{A} \vec{f} = {}^t\mathbf{R} \cdot \vec{p} = \vec{q} \Leftrightarrow \vec{f} = \arg \min_{f \in C} \|\vec{p} - \mathbf{R} \vec{f}\|^2$
- R injectif ? : choix parmi les solutions possibles
- \mathbf{R}^{-1} continue mais $\|\mathbf{R}^{-1}\|$ grande : $\kappa(\mathbf{R}) = \|\mathbf{R}\| \|\mathbf{R}^{-1}\| = \frac{\mu_{\max}}{\mu_{\min}} \gg 1$

$$\frac{\|\delta \vec{f}\|}{\|\vec{f}\|} \leq \frac{\kappa(\mathbf{R})}{1 - \kappa(\mathbf{R}) \frac{\|\delta \mathbf{R}\|}{\|\mathbf{R}\|}} \left[\frac{\|\delta \vec{p}\|}{\|\vec{p}\|} + \frac{\|\delta \mathbf{R}\|}{\|\mathbf{R}\|} \right]$$

Estimation de $\kappa(R)$

◆ Gradient conjugué

$$\omega^j = \frac{\|\vec{r}^j\|^2}{\langle \vec{d}^j | R^* \cdot R \cdot \vec{d}^j \rangle} \quad \beta^j = \frac{\|\vec{r}^{j+1}\|^2}{\|\vec{r}^j\|^2}$$

◆ Approximation de Lanczos-Galerkin

$$\begin{pmatrix} \frac{1}{\omega^0} & -\frac{\sqrt{\beta^0}}{\omega^0} & 0 & 0 \\ -\frac{\sqrt{\beta^0}}{\omega^0} & \frac{1}{\omega^1} + \frac{\beta^0}{\omega^0} & \ddots & 0 \\ 0 & \ddots & \ddots & -\frac{\sqrt{\beta^{j-1}}}{\omega^{j-1}} \\ 0 & 0 & -\frac{\sqrt{\beta^{j-1}}}{\omega^{j-1}} & \frac{1}{\omega^j} + \frac{\beta^{j-1}}{\omega^{j-1}} \end{pmatrix}$$

◆ dont le spectre CV vers celui de R: estimation de $\kappa(R)$

Exploitation du spectre

- ◆ Estimation du conditionnement $N = \frac{\lambda_{\max}}{\lambda_{\min}}$
 - ◆ Ajustement d'un paramètre (ex: fréquence de coupure) en fonction du conditionnement

◆ Majoration des erreurs

- ◆ Permet de fixer un critère d'arrêt des itérations

- ◆ Sur les coupes $\varepsilon^j = \frac{1}{\lambda_{\max}} \frac{\|\vec{r}^j\|}{\|\vec{f}^j\|}$
- ◆ Sur le spectre $|\lambda - \tilde{\lambda}| \leq \eta(\tilde{\lambda}) = \frac{\sqrt{\beta_{i+1}} \omega_{i+1} \tilde{v}_{der}^{\lambda,i}}{\tilde{\lambda}_{\max}^i \|\tilde{v}^{\lambda,i}\|}$

Régularisation

Remplacer : $\bar{f} = \arg \min_{f \in C} \|\vec{p} - R\vec{f}\|^2$ par

$$\bar{f} = \arg \min_{f \in C} \left\{ \|\vec{p} - R\vec{f}\|^2 + \alpha \cdot \rho(\vec{f}) \right\}$$

Adéquation aux données
Surjectivité du problème inverse

Régularisation
injectivité

Exemples : $\rho(\vec{f}) \in \left\{ \|\vec{f}\|^2 ; \langle \vec{f} | Q\vec{f} \rangle ; \sum_i f_i \ln(f_i) ; \dots \right\}$

Régularisation de Tikhonov

(cf. pseudo-inverse de Moore-Penrose)

$$\vec{f} = \arg \min_{f \in C} \left\{ \|\vec{p} - R\vec{f}\|^2 + \alpha \|\vec{f}\|^2 \right\}$$

Adéquation aux données
Surjectivité du problème inverse

Régularisation
injectivité

$$\vec{f} = \arg \min_{\vec{f}} \left\{ \|\vec{p} - R\vec{f}\|^2 + \alpha \|\vec{f}\|^2 \right\} \Leftrightarrow (R^t R + \alpha I) f = R^t p$$

$$\vec{f} = (R^t R + \alpha I)^{-1} R^t p$$

Régularisation MAP-EM-OSL

Bayes : $P(\vec{f} / \vec{p}) = P(\vec{p}/\vec{f}) \cdot P(\vec{f}) / P(\vec{p}) = P(\vec{p}/\vec{f}) \cdot P(\vec{f})$

$$\vec{f} \approx \arg \min_{\vec{f}} \left[-\log P(\vec{p}/\vec{f}) - \log P(\vec{f}) \right]$$

Adéquation aux données

$$\log \prod_i \frac{e^{-\tilde{p}_i} \tilde{p}_i^{p_i}}{p_i!}$$

?

Régularisation MAP-EM-OSL

Bayes : $P(\vec{f} / \vec{p}) = P(\vec{p}/\vec{f}) \cdot P(\vec{f}) / P(\vec{p}) = P(\vec{p}/\vec{f}) \cdot P(\vec{f})$

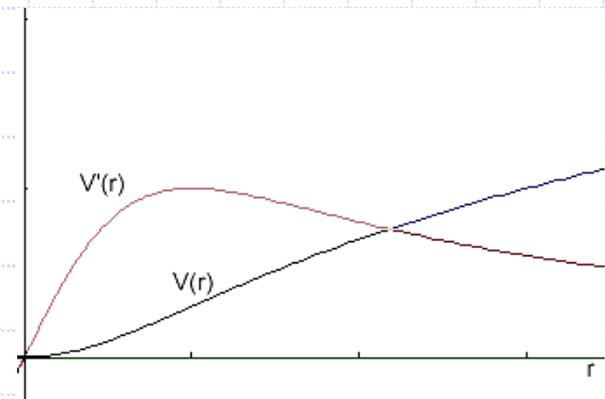
$$\vec{f} \approx \arg \min_{\vec{f}} \left[-\log P(\vec{p}/\vec{f}) - \log P(\vec{f}) \right]$$

Adéquation aux données

$$\log \prod_i \frac{e^{-\tilde{p}_i} \tilde{p}_i^{p_i}}{p_i!}$$

$$P(\vec{f}) = \frac{1}{K} e^{-\beta \cdot \sum_{i,j} w_{i,j} \cdot V(f_i - f_j)}$$

Distribution de Gibbs

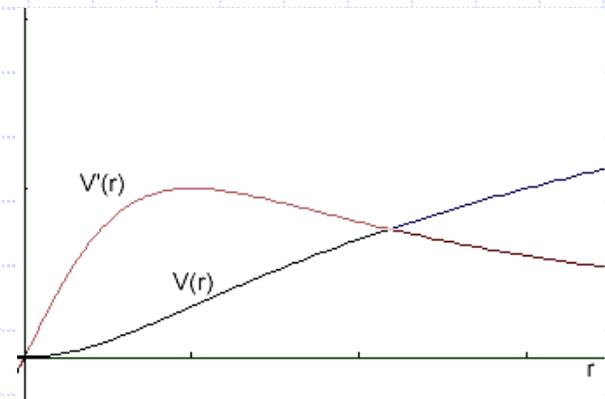


Régularisation MAP-EM-OSL

$$\tilde{\vec{f}} = \arg \min_{\vec{f}} \left[-\log P(\vec{p}/\vec{f}) + \beta \sum_{i,j} w_{i,j} V(f_i - f_j) \right]$$

$$f_i^{n+1} = f_i^n \cdot \frac{1}{\sum_{l'=1}^P r_{l',i} + \beta \cdot \partial U} \cdot \sum_{l=1}^P r_{l,i} \frac{p_l}{\sum_{s=1}^N r_{l,s} f_s^n}$$

$$\partial U = \sum_{f_k \in V(f_i)} w_{i,k} \cdot \frac{\partial V}{\partial r}(f_i - f_k)$$



Régularisation de Fourier FRECT

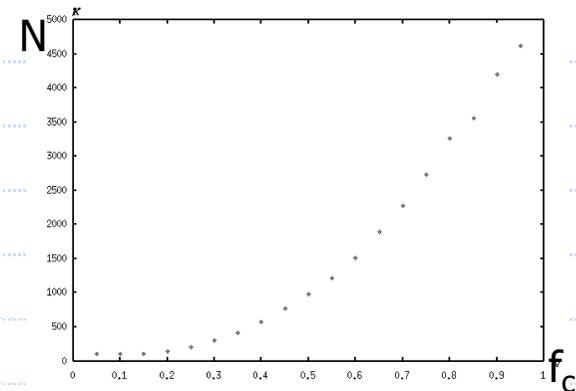
à régulariser

$$\hat{f} = B.\hat{f} + (1 - B).\hat{f}$$

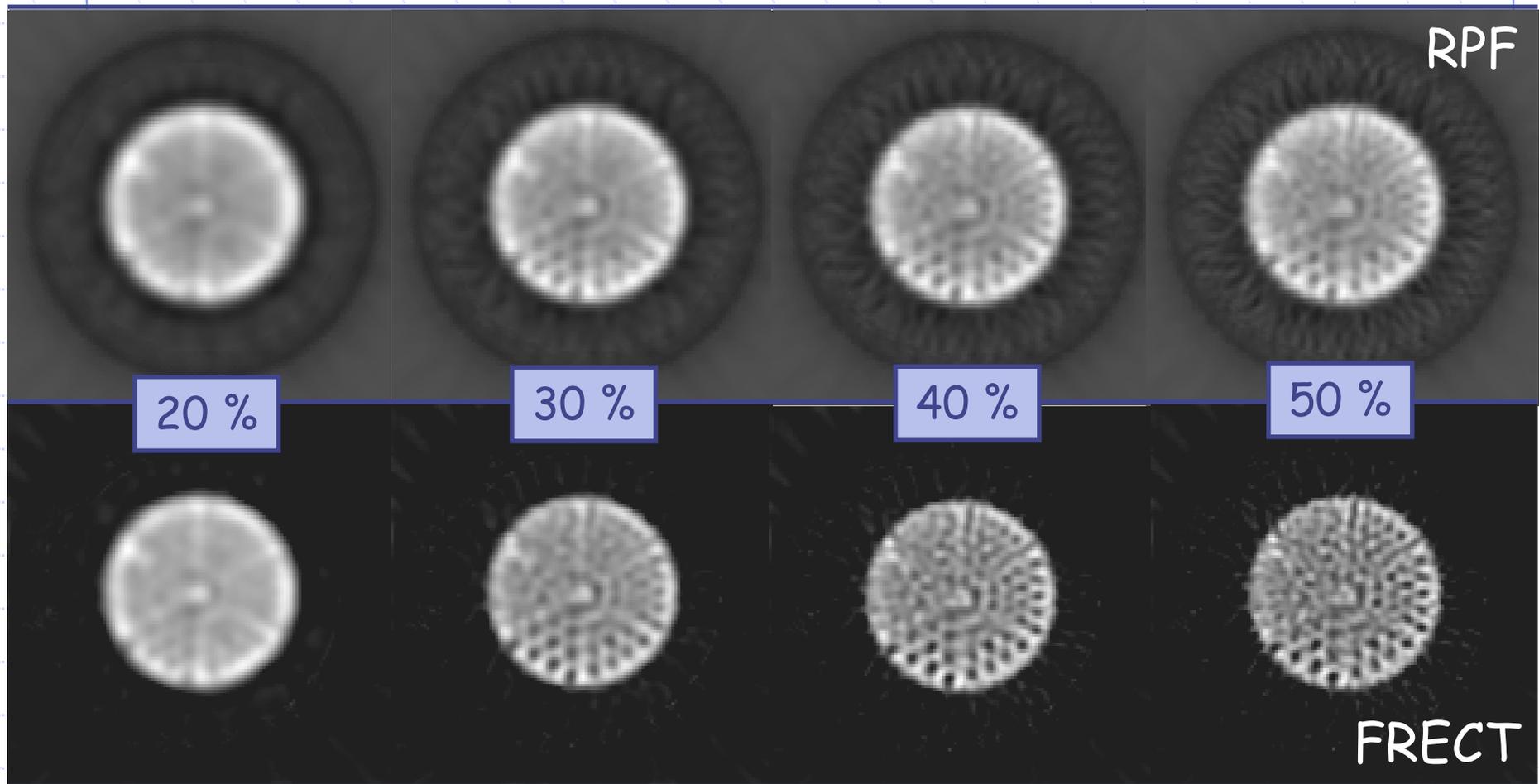
Adéquation à $p' = R.f'$

$$\hat{p}' = \hat{f}' = B.\hat{f} = B.\hat{p} \Rightarrow p' = TF_s^{-1} B.\hat{p}$$

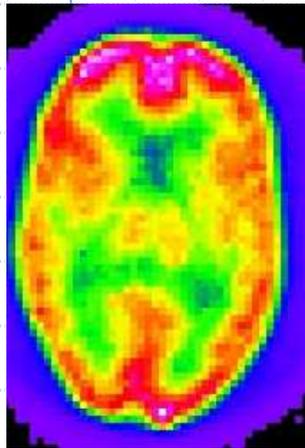
$$E(f) = \left\| TF_s^{-1}(B.\hat{p}) - Rf \right\|^2 + \left\| (1 - B).\hat{f} \right\|^2$$



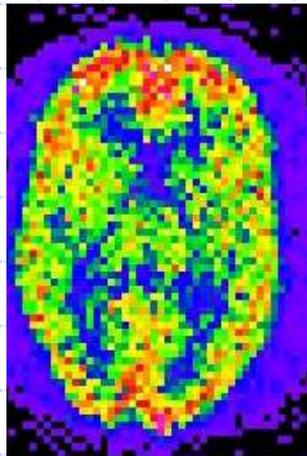
Comparaison RPF-FRECT



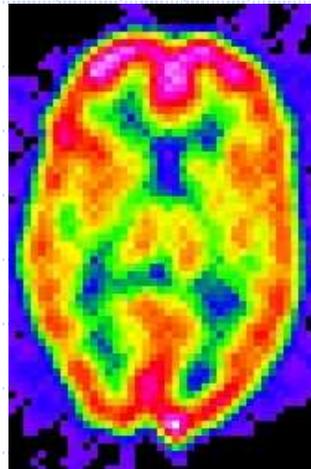
Comparaison MLEM-GC-FRECT



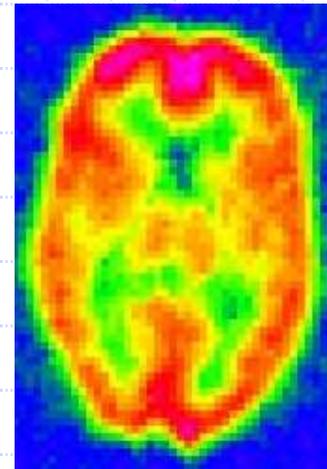
MLEM 6



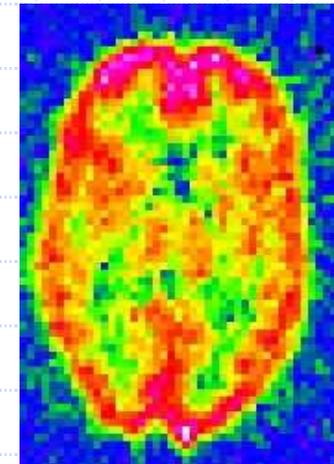
MLEM 200



FRECT 34 (CV)



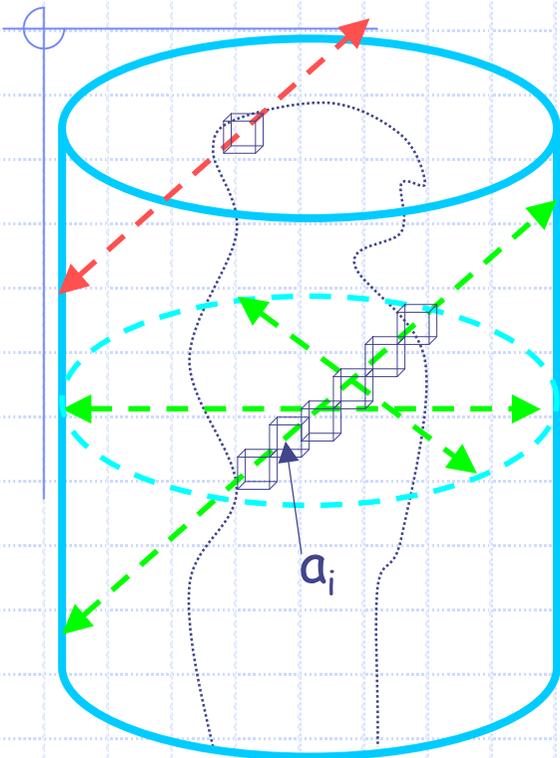
GC 6



GC 16

$$E(f) = \left\| TF_s^{-1}(B \cdot \hat{p}) - Rf \right\|^2 + \left\| (1 - B) \cdot \hat{f} \right\|^2$$

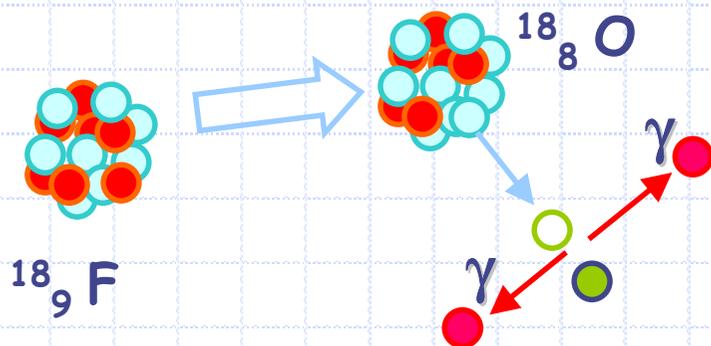
Tomographie en coïncidence 3D



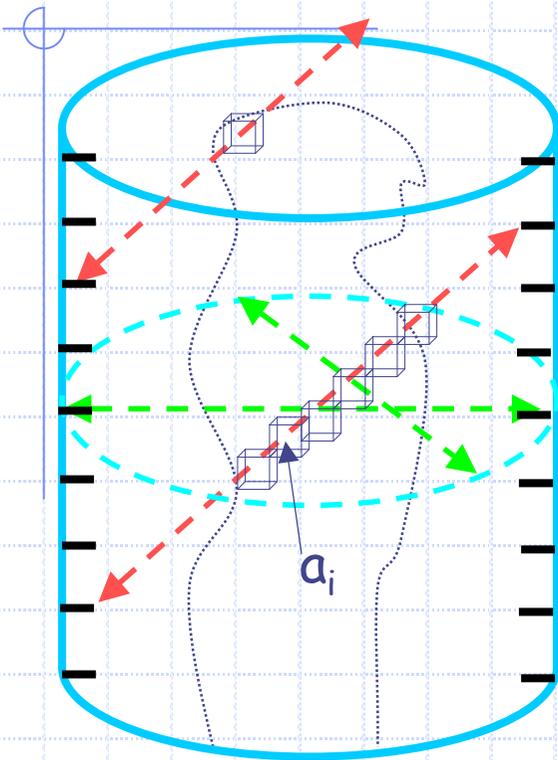
Exemple de la TEP

Projections 3D **redondantes** et **incomplètes**

- Recherche de $f(x,y,z)$ connaissant $p(s,\phi,z,\theta)$
- Certaines projections obliques ne sont pas enregistrées si $\theta \neq 0$

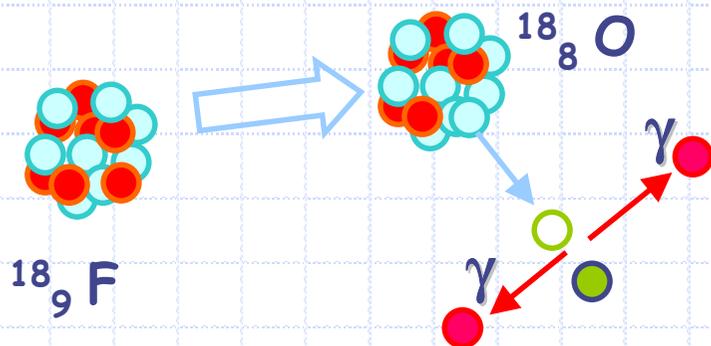


Tomographie en coïncidence 3D

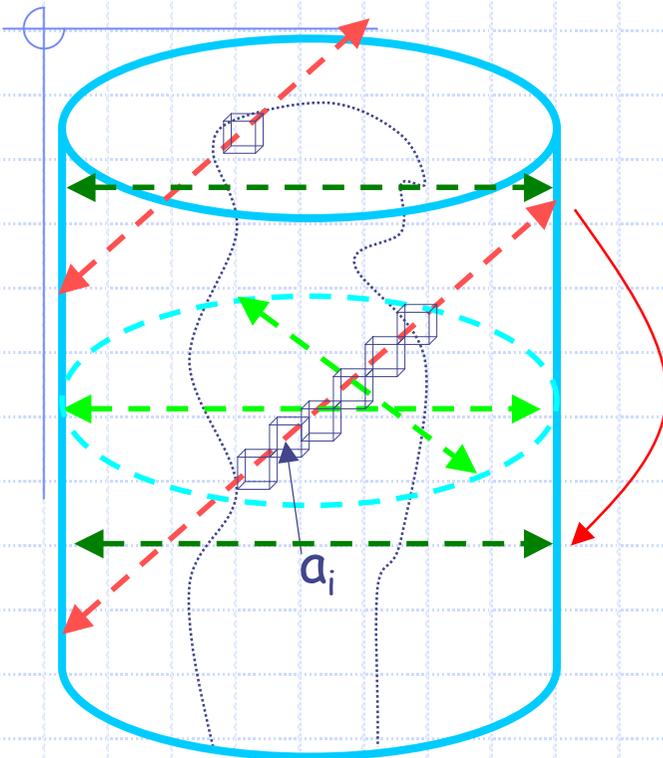


Projections 3D **redondantes** et **incomplètes**

- Reconstruction 2D de données 2D
 - Utilisation d'un collimateur
 - \searrow statistique de comptage, \searrow S/B

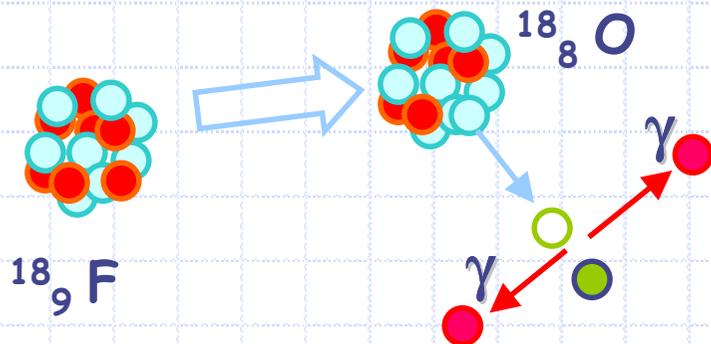


Tomographie en coïncidence 3D

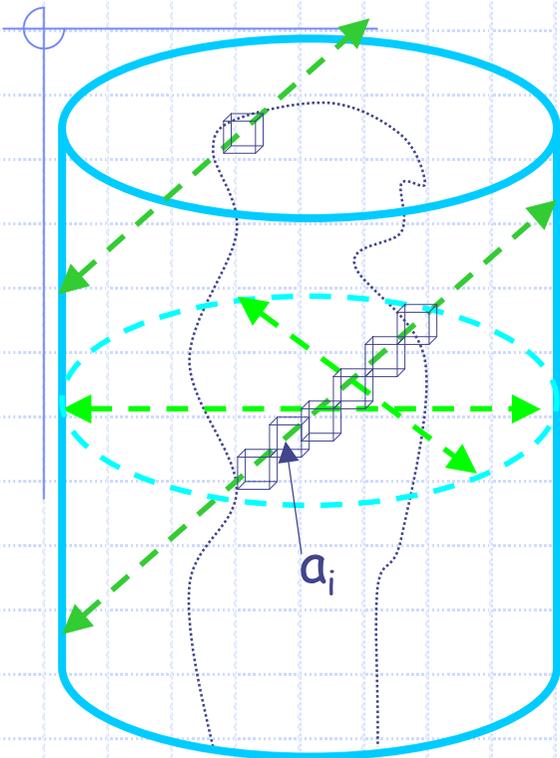


Projections 3D **redondantes** et **incomplètes**

- Reconstruction 2D de données 2D
 - Utilisation d'un collimateur
 - $\sphericalangle \gamma$ détectés N , $\sphericalangle S/B = N/\sqrt{N} = \sqrt{N}$
- Réarrangement 2D de données 3D
 - Algorithmes de «rebinning»
 - $S/B \nearrow$ mais approximation

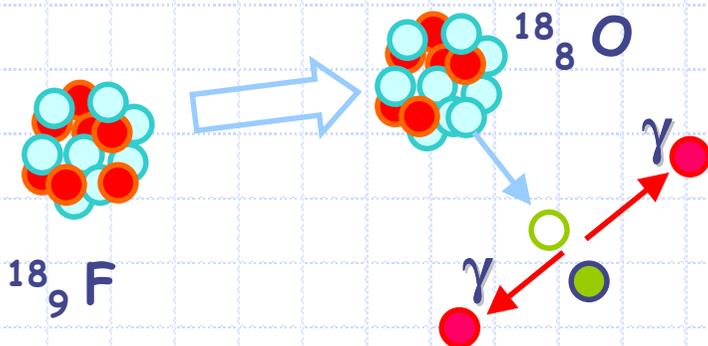


Tomographie en coïncidence 3D



Projections 3D **redondantes** et **incomplètes**

- Reconstruction 2D de données 2D
 - Utilisation d'un collimateur
 - $\sphericalangle \gamma$ détectés N , $\sphericalangle S/B=N/\sqrt{N}=\sqrt{N}$
- Réarrangement 2D de données 3D
 - Algorithmes de «rebinning»
 - $S/B \nearrow$ mais approximation
- Reconstruction 3D de données 3D
 - Algorithmes algébriques 3D
 - RPF 3D si projections complètes
 - $S/B \nearrow$ mais temps de calcul $\nearrow \nearrow$



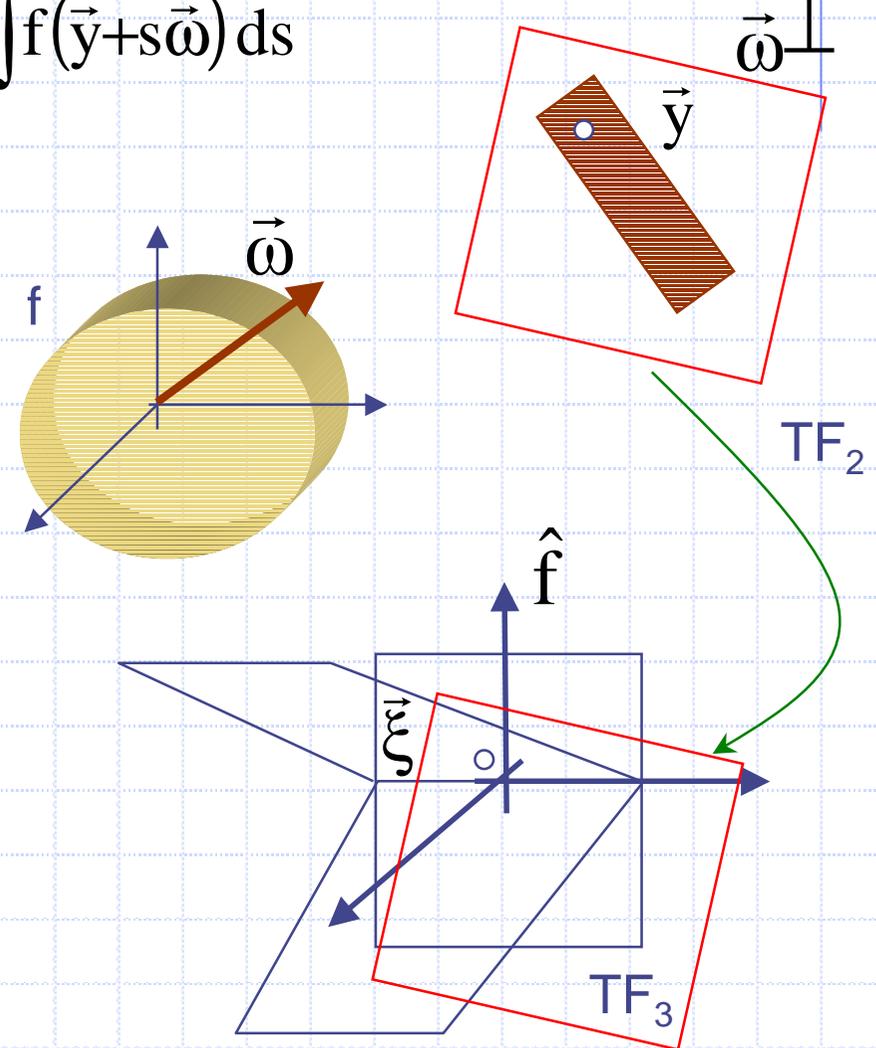
Un théorème de Radon 3D...

$$\forall \vec{\omega} \in S, \quad \forall \vec{y} \in \vec{\omega}^\perp, p_{\vec{\omega}}(\vec{y}) = \int f(\vec{y} + s\vec{\omega}) ds$$

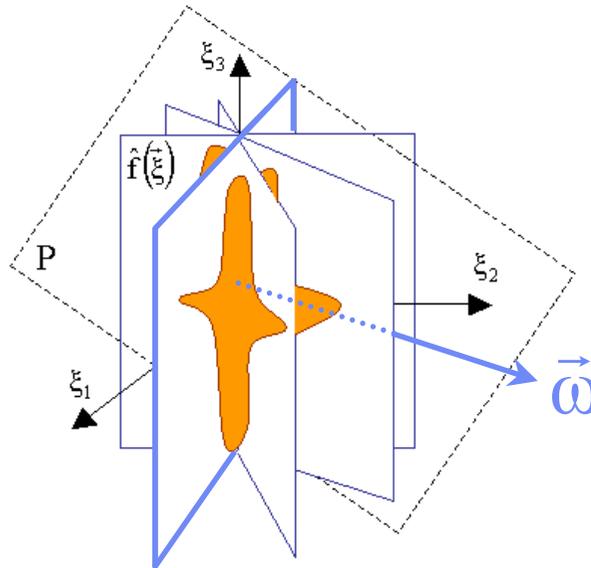
$$\hat{p}_{\vec{\omega}}(\vec{\xi}) = \iint_{\vec{\omega}^\perp} \int f(\vec{y} + s\vec{\omega}) e^{-2i\pi \vec{y} \cdot \vec{\xi}} ds d\vec{y}$$

$$\hat{p}_{\vec{\omega}}(\vec{\xi}) = \iiint f(\vec{x}) e^{-2i\pi \vec{x} \cdot \vec{\xi}} d\vec{x} = \hat{f}(\vec{\xi})$$

$$\forall \vec{\xi} \in \vec{\omega}^\perp, \hat{p}_{\vec{\omega}}(\vec{\xi}) = \hat{f}(\vec{\xi})$$



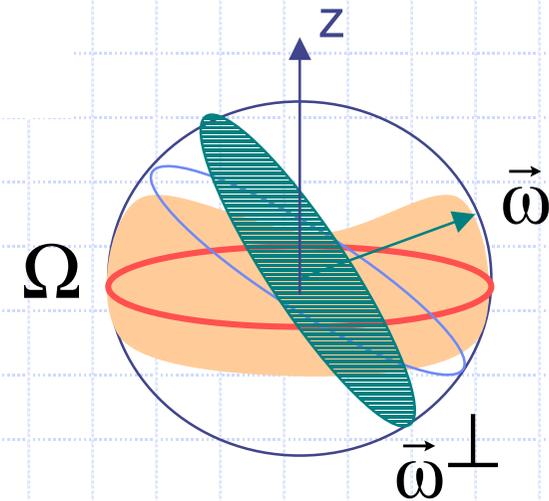
1- Condition d'Orlov :



Condition nécessaire à l'affectation de toutes les fréquences spatiales de \mathbb{R}^3 :

Ω contient au moins un cercle équatorial de S
i.e

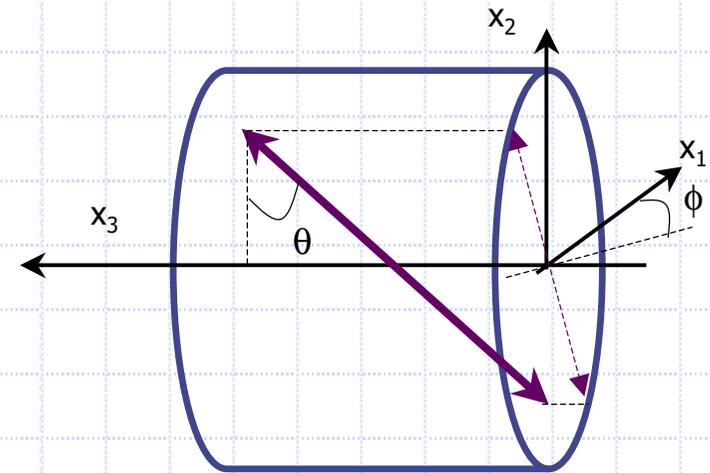
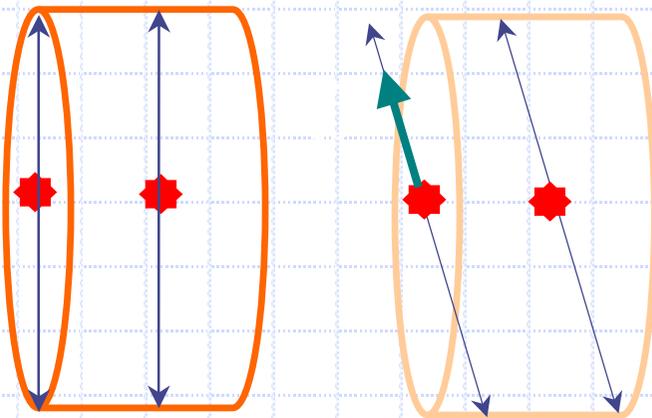
Ω intersecte tout cercle équatorial de S



... plutôt difficile à appliquer !

1- Condition d'Orlov

2 - Projections non tronquées



3 – moyennant une interpolation 3D dans le domaine des fréquences

$$\hat{p}(\xi_1, \xi_2) = \hat{f}(\xi_1 \cos \theta \sin \phi - \xi_2 \sin \theta, \xi_1 \sin \theta \sin \phi + \xi_2 \cos \theta, -\xi_1 \cos \phi)$$

Solutions possibles

1 – Condition d'Orlov

- Détecteur TEP cylindrique

2 – Projections tronquées

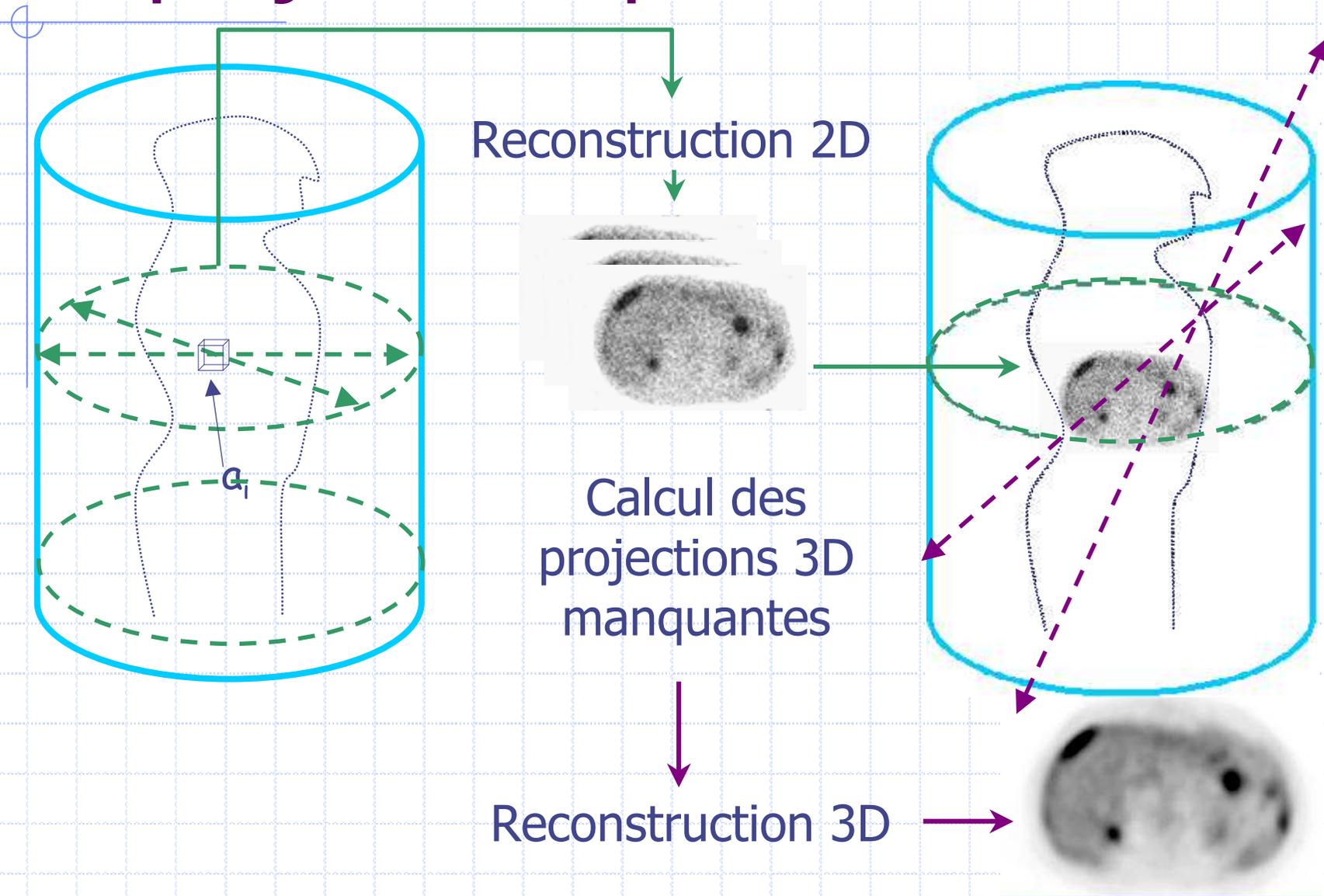
- Estimées par reconstruction 2D puis projection ou rebinning

3 – Interpolation 3D en fréquence

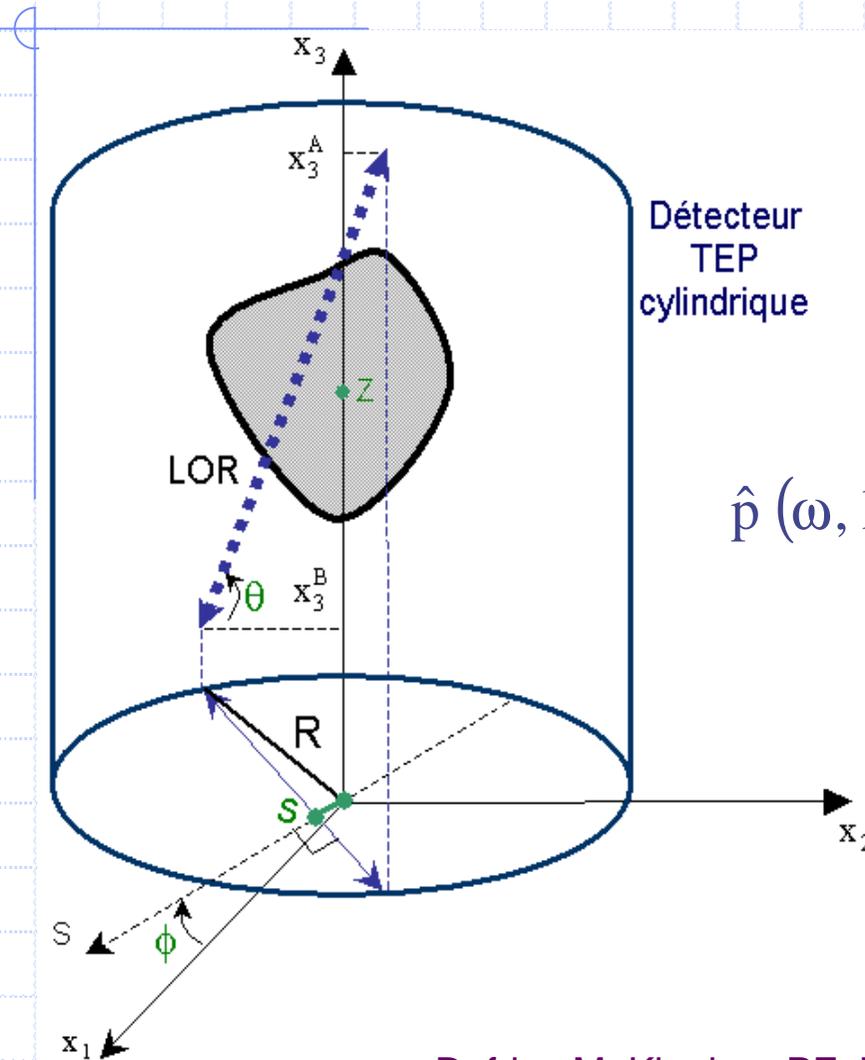
- Optimisation de l'interpolation (fonctions de Kaiser-Bessel)
- Utilisation d'une rétro-projection filtrée (filtre de Colsher)

En « routine » : Utilisation d'algorithmes algébriques (OSEM 3D)
Reconstruction 2D après rebinning des projections 3D

Reprojection après RPF 2D



Ré-arrangement (rebinning) exact



$$p \left(s, \phi, z = \frac{x_3^A + x_3^B}{2}, \delta = tg \theta \right)$$

TF(s,phi) puis TF(z)
si invariance en Tz

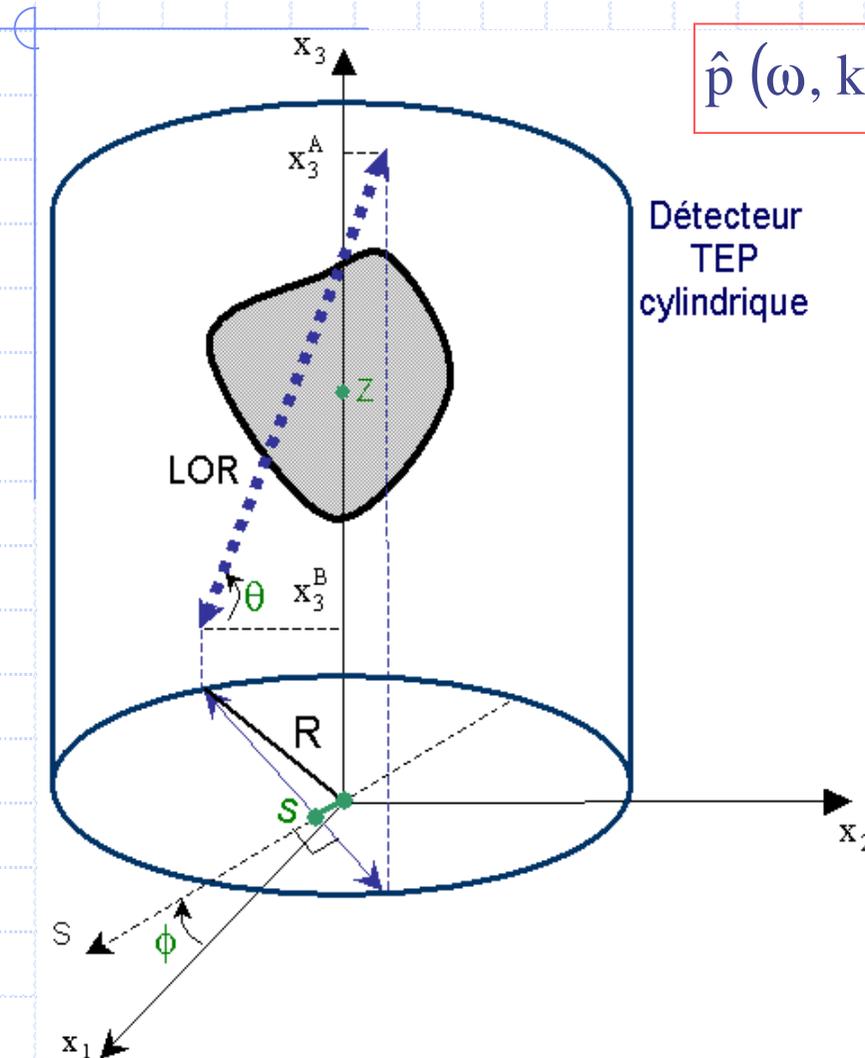
$$\hat{p}(\omega, k, \zeta, \delta) = e^{-ik \arctan(\alpha)} \hat{p}(\omega\sqrt{1+\alpha^2}, k, \zeta, 0)$$

$$\alpha = \frac{\delta\zeta}{\omega}$$

Defrise M, Kinahan PE, Townsend DW, Michel C, Sibomana M, Newport DF.

Exact and approximate rebinning algorithms for 3-D PET data. IEEE Trans Med Imaging 1997;16:145-58.

Ré-arrangement approximatif



$$\hat{p}(\omega, k, \zeta, \delta) = e^{-ik \arctan(\alpha)} \hat{p}(\omega \sqrt{1+\alpha^2}, k, \zeta, 0)$$

DL à l'ordre 1 sur $\alpha = \frac{\delta \zeta}{\omega}$

$$\hat{p}(\omega, k, \zeta, \delta) \approx e^{-ik\alpha} \hat{p}(\omega, k, \zeta, 0)$$

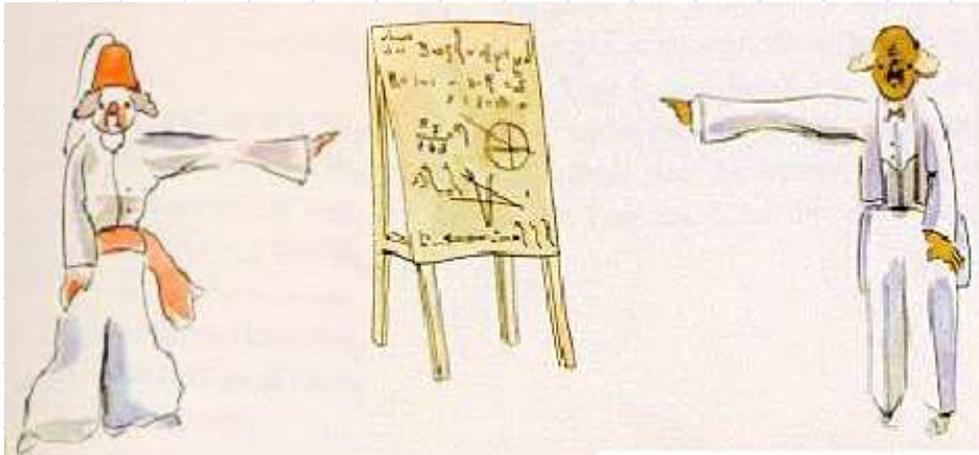
$$\hat{p}(\omega, k, z, \delta) \approx \hat{p}\left(\omega, k, z - k \frac{\delta}{\omega}, 0\right)$$

SYNTHESE DE DONNEES 2D à S/B \uparrow :

$$\hat{p}(\omega, k, z, 0) \approx \hat{p}\left(\omega, k, z + k \frac{\delta}{\omega}, \delta\right)$$

SYNTHESE DE DONNEES MANQUANTES:

$$\hat{p}(\omega, k, z, \delta) \approx \hat{p}\left(\omega, k, z - k \frac{(\delta - \delta')}{\omega}, \delta'\right)$$



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35-105-A-10, 2009.

Merci de votre attention...